A Posteriori Parameter Selection for Local Regularization of Nonlinear Volterra Equations of Hammerstein Type

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Inverse, Ill-posed Problems

Let $\mathcal{A} : X \to X$ be an operator, X a Banach space.

Consider solving

$$\mathcal{A}u = f$$

for $u \in dom(\mathcal{A})$, where the *data* $f \in X$ and $\mathcal{A}$ are known.

In this talk,
$\mathcal{A} : C[0,1] \to C[0,1]$ is a nonlinear Volterra Hammerstein convolution operator:

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$$\mathcal{A}u(t) = \int_0^t k(t-s)g(s,u(s))\,ds = f(t), \quad t \in [0,1],$$

for $\bar{u} \in C[0,1]$, given data $f \in Range(\mathcal{A})$, kernel $k \in C^{\nu}([0,1])$ is ν-smoothing, and nonlinear function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous.

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$$u(t) \rightarrow g(t, u(t)) \rightarrow \int_0^t k(t-s)g(s, u(s))ds$$

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Ill-posedness of the problem means solution to $\mathcal{A}u = f^\delta$ is a poor approximation.

Approach

- Use a *regularization* scheme on $\mathcal{A}u = f$. Typically solve a family of "nearby" parameter-dependent *well-posed* equations, $R_\alpha u = f$ whose solutions, u_α,
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  - $\|u_\alpha^\delta - \bar{u}\| \to 0$ as $\delta \to 0$.

Approximating Equation (Local Regularization)

One-step equation (Luo ('07), Brooks, Lamm, and Luo ('10))
Approximate $\bar{u}$ by solving:

$$a_\alpha \mathcal{G} u + \mathcal{A}_\alpha u = f_\alpha,$$

where

$$\begin{aligned} a_\alpha &= \int_0^\alpha \int_0^\rho k(\rho - s) ds \; d\eta_\alpha(\rho) \\ f_\alpha &= \int_0^\alpha f(t+\rho) d\eta_\alpha(\rho) \end{aligned}$$

$$\begin{aligned} \mathcal{A}_\alpha w(t) := \int_0^t \int_0^\alpha & k(t+\rho-s) d\eta_\alpha(\rho) \; g(s, w(s)) \; ds \\ & - a_\alpha g(t, w(t)) + a_\alpha g(t, w(t-\tau)) \\ & + a_\alpha g_x(t, w(t-\tau)) \left(w(t) - w(t-\tau) \right), \end{aligned}$$

for $\tau = \tau(t, \alpha)$ suitably chosen.

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$$\left\| u^\delta_{\alpha(\delta)} - \bar{u} \right\| = O\left(\delta^{\frac{\gamma}{\gamma+\nu}}\right) \text{ as } \delta \to 0.$$

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- Numerical implementation involves solving a nonlinear equation at only the first step (improvement over Two-step approach (Lamm and Dai 2005)).
- New results (2010) - Established well-posedness of the One-step approximating equation in the absence of the *a priori* assumption $\alpha = K\delta^{\frac{1}{\gamma+\nu}}$. This was accomplished treating the equation as a perturbation of the Two-step equation (Lamm and Dai 2005) and making an appropriate choice of τ dependent on smoothness of data f^δ.

General Approximation Error and Parameter Selection Strategy

Consider the bound

$$\left\| u_\alpha^\delta - \bar{u} \right\| \leq \| u_\alpha - \bar{u} \| + \left\| u_\alpha^\delta - u_\alpha \right\| .$$

- Error due to regularization: $\| u_\alpha - \bar{u} \| \to 0$ as $\alpha \to 0$.
- Error due to regularization and noise in the data:

$$\left\| u_{\alpha(\delta)}^\delta - u_\alpha \right\| \to \infty \text{ as } \alpha \to 0.$$

Choose the regularization parameter $\alpha = \alpha(\delta)$ (*a posteriori*) leading to convergent methods, i.e.

1. $\alpha(\delta) \to 0$ as $\delta \to 0$.
2. $\left\| u_\alpha^\delta - \bar{u} \right\| \to 0$ as $\delta \to 0$.

A Posteriori Parameter Selection Strategies

For $\tau \in (1,2)$ fixed, choose α for which

- **Morozov's Discrepancy Principle**

$$d(\alpha) = \left\| \mathcal{A}u_{\alpha}^{\delta} - f^{\delta} \right\| = \tilde{\tau}\delta.$$

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- **(Modified) Discrepancy Principle for Local Regularization with One-step equation**

$$d(\alpha) = a_\alpha^m \left\| \mathcal{A}_\alpha u_\alpha^\delta - f_\alpha^\delta \right\| = \tilde{\tau}\delta, \quad m > 0$$

where $\mathcal{A}_\alpha$ and f_α^δ as as defined above for One-step local regularizing equation for the Hammerstein problem.

A posteriori Convergence

Theorem

Under suitable conditions on η_α and the linearization parameter τ, it can be shown that, for $\delta > 0$ sufficiently small or for $\|f^\delta\|/\delta$ sufficiently large, there exists a smallest $\alpha = \alpha(\delta) \in (0, \bar{\alpha}]$ satisfying the Modified Discrepancy Principle for Local Regularization with the One-step equation. Further, if the selection of $\alpha(\delta)$ is made using the Principle, it follows that $\alpha(\delta) \to 0$ and $\|u_\alpha^\delta - \bar{u}\| \to 0$ as $\delta \to 0$.

Numerical Example

Solve

$$\int_0^t \frac{(t-s)^2}{2} u^3(s)ds = f(t), \quad t \in [0,1]$$

True solution $\bar{u}(t) = 8(t-0.4)^2 + 1$

A Priori Example

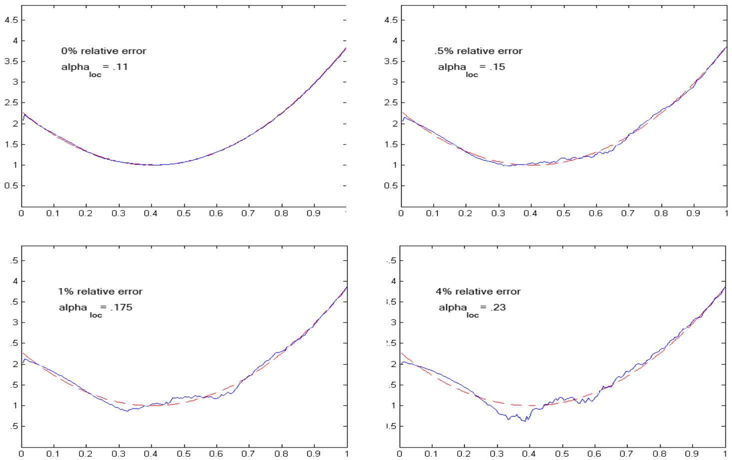

Figure: Three-smoothing problem with $0\%, 0.5\%, 1\%$ and 4% relative error in the data. The value of $\alpha(\delta)$ is selected using that which minimizes the solution error.

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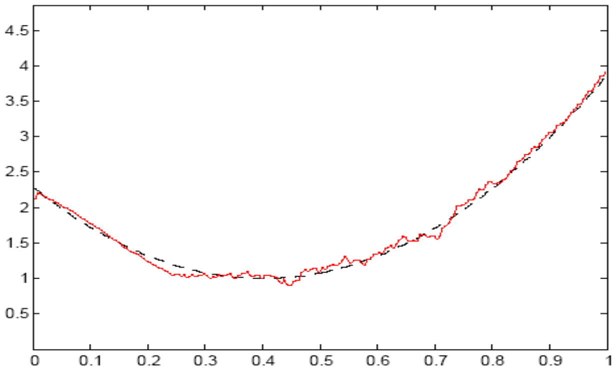

Figure: Three-smoothing problem with 1% relative error in the data ($\delta = 0.0067$). We use $m = .01$ and $\tilde{\tau} = \sqrt{2}$ and $\bar{\alpha} = .25$. The value of $\alpha(\delta) = 0.165$ is selected using the modified discrepancy principle; for u_α^δ determined using this α, there is 4.4% relative error in the recovered solution.